EMiDO: Blanchard-Kahn method.

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Intro

- Method for solving linear rational expectations models.
- First eigenvalues.
- Solving linear difference equations.
- Solving system of linear difference equations.
- Blanchard-Kahn method

Matrix algebra

Eigenvectors and Eigenvalues

- Consider an $n \times n$ square matrix A.
- λ is an eigenvalue of A and x is an eigenvector of A if they satisfy the equation

  $$Ax = \lambda x$$

  where x is a non-zero vector.
- To find an eigenvalues we can rewrite the above equation

  $$\begin{aligned} Ax - \lambda x &= 0 \\ (A - \lambda I)x &= 0 \end{aligned}$$

  where I is an identity matrix.
- If $[A - \lambda I]$ is invertible, then $x = [A - \lambda I]^{-1} \cdot 0$ is a vector of zeros. To have a non-zero solution for x a matrix $[A - \lambda I]$ needs to be non-invertible, i.e. singular.
- For $[A - \lambda I]$ to be non-invertible it needs to have a zero determinant (a square matrix is singular if and only if its determinant is 0)

  $$\det(A - \lambda I) = 0$$

  and we can use this formula to find eigenvalues.
- Diagonalization (Jordan decomposition). Eigenvalues and eigenvectors can be used to diagonalize a matrix. An $n \times n$ matrix A is diagonalizable if we can express the matrix as the product of an invertible square matrix C and a diagonal matrix so that $A = C\Lambda C^{-1}$, where
  - Λ is a diagonal matrix with the eigenvalues of A on the diagonal.
  - the ith column of matrix C is an eigenvector corresponding to the ith eigenvalue of A.
- No, an $n \times n$ matrix A is diagonalizable if and only if its eigenvectors are linearly independent.

Example 1.

- Consider a matrix $A = \begin{bmatrix} 2 & 4 \\ 1 & -1 \end{bmatrix}$, then

$$\begin{aligned} A - \lambda I &= \begin{bmatrix} 2 & 4 \\ 1 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 4 \\ 1 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \\ &= \begin{bmatrix} 2-\lambda & 4 \\ 1 & -1-\lambda \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \det(A - \lambda I) = (2-\lambda)(-1-\lambda) - 4 &= 0 \\ -2 - 2\lambda + \lambda + \lambda^2 - 4 &= 0 \\ \lambda^2 - \lambda - 6 &= 0 \\ (\lambda - 3)(\lambda + 2) &= 0 \end{aligned}$$

Therefore we have two eigenvalues $\lambda_1 = 3$ and $\lambda_2 = -2$.

- Since eigenvectors are the solution to $[A - \lambda I]x = 0$ and we have to eigenvalues we have to eigenvectors. Start with $\lambda_1 = 3$

$$\begin{aligned} \begin{bmatrix} 2-\lambda & 4 \\ 1 & -1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= 0 \\ \begin{bmatrix} 2-3 & 4 \\ 1 & -1-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= 0 \\ \begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= 0 \end{aligned}$$

So

$$\begin{aligned} -x_1 + 4x_2 &= 0 \\ x_1 - 4x_2 &= 0 \end{aligned}$$

So the eigenvector is $\begin{bmatrix} 4a \\ a \end{bmatrix}$.

- And for $\lambda_2 = -2$

$$\begin{aligned} \begin{bmatrix} 2-\lambda & 4 \\ 1 & -1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= 0 \\ \begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= 0 \end{aligned}$$

$$\begin{aligned} x_1 + x_2 &= 0 \\ 4x_1 + 4x_2 &= 0 \end{aligned}$$

So the eigenvector is $\begin{bmatrix} a \\ -a \end{bmatrix}$.

- We are going to chose an eigenvectors such that one of the terms is set to 1, therefore we get two eigenvectors $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ (corresponding to $\lambda_1 = 3$) and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ (corresponding to $\lambda_2 = -2$).

- For a matix $A = \begin{bmatrix} 2 & 4 \\ 1 & -1 \end{bmatrix}$, with eigenvalues $\lambda_1 = 3$ and $\lambda_2 = -2$ and eigenvectors $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ (corresponding to $\lambda_1 = 3$) and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ (corresponding to $\lambda_2 = -2$) we have

$$\Lambda = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$$

and

$$C = \begin{bmatrix} 4 & 1 \\ 1 & -1 \end{bmatrix}$$

we have

$$C^{-1} = \begin{bmatrix} 0.2 & 0.2 \\ 0.2 & -0.8 \end{bmatrix}$$

- Note

$$\begin{aligned} C\Lambda C^{-1} &= \begin{bmatrix} 4 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 0.2 & 0.2 \\ 0.2 & -0.8 \end{bmatrix} \\ &= \begin{bmatrix} 12 & -2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 0.2 & 0.2 \\ 0.2 & -0.8 \end{bmatrix} \\ &= \begin{bmatrix} 12 \cdot 0.2 - 2 \cdot 0.2 & 12 \cdot 0.2 + 2 \cdot 0.8 \\ 3 \cdot 0.2 + 2 \cdot 0.2 & 3 \cdot 0.2 - 2 \cdot 0.8 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 4 \\ 1 & -1 \end{bmatrix} \\ &= A \end{aligned}$$

Difference equations

Solution method

- Consider a simple difference equation

$$x_t = \alpha x_{t-1} + \beta$$

- First we find the steady state

$$\begin{aligned} x - \alpha x &= \beta \\ x &= \frac{\beta}{1-\alpha} \end{aligned}$$

- Let $\tilde{x}_t \equiv x_t - x$ (which implies $x_t = x + \tilde{x}_t$) then

$$\begin{aligned} x + \tilde{x}_t - \alpha(x + \tilde{x}_{t-1}) - \beta &= 0 \\ \tilde{x}_t + \alpha \tilde{x}_{t-1} &= 0 \end{aligned}$$

Guess

$$\tilde{x}_t = A\lambda^t$$

Substituting back

$$\begin{aligned} \tilde{x}_t - \alpha \tilde{x}_{t-1} &= 0 \\ A\lambda^t - \alpha A \lambda^{t-1} &= 0 \\ \lambda - \alpha &= 0 \\ \lambda &= \alpha \end{aligned}$$

and

$$\tilde{x}_t = A\alpha^t$$

Therefore

$$x_t = x + A\alpha^t$$

where $x = \frac{\beta}{1-\alpha}$. To find A we need at least one value of x_t, usually in economic applications we know the initial conditions x_0 (sometimes it is also a terminal condition), then

$$\begin{aligned} x_0 &= \frac{\beta}{1-\alpha} + A\alpha^0 \\ A &= x_0 - \frac{\beta}{1-\alpha} \end{aligned}$$

- So the solution is

$$\begin{aligned} x_t &= \frac{\beta}{1-\alpha} + (x_0 - \frac{\beta}{1-\alpha})\alpha^t \\ x_t &= x_0\alpha^t + \frac{\beta}{1-\alpha}(1-\alpha^t) \\ x_t &= x_0\alpha^t + \beta\frac{1-\alpha^t}{1-\alpha} \end{aligned}$$

Note that method works for $|\alpha| \neq 1$.

Stability

- The above equation is stable when $|\alpha| < 1$, in which case

$$\lim_{t\to\infty} x_t = \lim_{t\to\infty} (x_0\alpha^t + \beta\frac{1-\alpha^t}{1-\alpha}) = \frac{\beta}{1-\alpha}$$

$$x_t = \alpha x_{t-1} + \beta$$

- The above equation is unstable when $|\alpha| > 1$, in which case (technically speaking sometimes there may be no limit becasue one subsequence converges to $+\infty$ and the other to $-\infty$)

$$\lim_{t\to\infty} x_t = \lim_{t\to\infty} (x_0\alpha^t + \beta\frac{1-\alpha^t}{1-\alpha}) = +\infty \text{ or } -\infty$$

Example 2.

- Consider a simple difference equation

$$x_t - \alpha x_{t-1} - 100 = 0$$

i.e. $\beta = 100$.

- First we find the steady state

$$\begin{aligned} x - \alpha x &= 100 \\ x &= \frac{100}{1-\alpha} \end{aligned}$$

- Let $\tilde{x}_t \equiv x_t - x$ (which implies $x_t = x + \tilde{x}_t$) then

$$\begin{aligned} x + \tilde{x}_t - \alpha(x + \tilde{x}_{t-1}) - 100 &= 0 \\ \tilde{x}_t + \alpha\tilde{x}_{t-1} &= 0 \end{aligned}$$

Guess

$$\tilde{x}_t = A\lambda^t$$

Substituting back

$$\begin{aligned}\tilde{x}_t - \alpha\tilde{x}_{t-1} &= 0\\ A\lambda^t - \alpha A\lambda^{t-1} &= 0\\ \lambda - \alpha &= 0\\ \lambda &= \alpha\end{aligned}$$

and

$$\tilde{x}_t = A\alpha^t$$

Therefore

$$x_t = x + A\alpha^t$$

where $x = \frac{100}{1-\alpha}$. To find A we need at least one value of x_t, usually in economic applications we know the initial conditions x_0 (sometimes it is also a terminal condition). Suppose $x_0 = 100$ then

$$\begin{aligned}x_0 = 100 &= \frac{100}{1-\alpha} + A\alpha^0\\ A &= 100 - \frac{100}{1-\alpha} = 100\frac{-\alpha}{1-\alpha}\end{aligned}$$

$x_t = x_0\alpha^t + \beta\frac{1-\alpha^t}{1-\alpha}$

- Consider different values of α:
  - $\alpha = 3$ then

$$\begin{aligned}x_t &= 100 + \frac{100}{1-3} - \frac{100}{1-3}\cdot 3^t\\ x_t &= 50 + 50\cdot 3^t\end{aligned}$$

    unstable, not oscillating (note $x_t > x$ for all t and $\lim_{t\to\infty} x_t = \infty$)
  - $\alpha = \frac{1}{2}$, then

$$\begin{aligned}x_t &= 100 + \frac{100}{1-0.5} - \frac{100}{1-0.5}\cdot\big(\frac{1}{2}\big)^t\\ x_t &= 300 - 300\cdot\big(\frac{1}{2}\big)^t\end{aligned}$$

    stable, not oscillating (note $x_t > x$ for all t and $\lim_{t\to\infty} x_t = x = 300$)
  - $\alpha = -\frac{1}{2}$, then

$$\begin{aligned}x_t &= 100 + \frac{100}{1-(-0.5)} - \frac{100}{1-(-0.5)}\cdot\big(-\frac{1}{2}\big)^t\\ x_t &= 166.67 - 66.7\cdot\big(-\frac{1}{2}\big)^2\end{aligned}$$

    stable, oscillating (note if $x_t > x$ then $x_{t+1} < x$ and $\lim_{t\to\infty} x_t = x = 166,67$)
  - $\alpha = -3$, then

$$\begin{aligned}x_t &= 100 + \frac{100}{1-(-3)} - \frac{100}{1-(-3)}\cdot\big(-3\big)^t\\ x_t &= 125 - 25(-3)^t\end{aligned}$$

    unstable, oscillating (note if $x_t > x$ then $x_{t+1} < x$ and $\lim_{t\to\infty} x_{2t} = -\infty$ and $\lim_{t\to\infty} x_{2t+1} = \infty$).
- Stability can be analyzed in phase diagrams.

System of difference equations

Solution method

- Consider a system of difference equations
$$x_t = Ax_{t-1} + b$$
where A is $n \times n$ matrix and b is an $n \times 1$ vector. We assume that A is diagonalizable.

- Since A is diagonalizable we can express it as $A = C\Lambda C^{-1}$.
$$x_t = C\Lambda C^{-1}x_{t-1} + b$$
Multiplying both sides by C^{-1} we get
$$\begin{aligned} C^{-1}x_t &= C^{-1}C\Lambda C^{-1}x_{t-1} + C^{-1}b \\ C^{-1}x_t &= \Lambda C^{-1}x_{t-1} + C^{-1}b \end{aligned}$$
Define $\bar{x}_t = C^{-1}x_t$ we get
$$\bar{x}_t = \Lambda\bar{x}_{t-1} + C^{-1}b$$
Since Λ is diagonal this becomes just a system of separate (independent) equations.

- To find the steady state of the system $x_t = Ax_{t-1} + b$ set $x_t = x_{t-1} = x$, which gives $x = [I - A]^{-1}b$.

- The stability of the system depends on the eigenvalues.

- If all the eigenvalues are less than 1 in absolute value then the system is stable. If all the eigenvalues are greater than 1 in absolute value then the system is unstable. If at least one eigenvalue is less than 1 in absolute value the system is saddle-path stable.

Example 3.

- Consider a simple difference equation
$$x_t = Ax_{t-1} + b$$
where $A = \begin{bmatrix} 2 & 4 \\ 1 & -1 \end{bmatrix}$ is taken from Example 1 and $b = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$. Let $x_0 = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$.

- Note it means
$$\begin{aligned} x_{1,t} &= 2x_{1,t-1} + 4x_{2,t-1} + 10 \\ x_{2,t} &= x_{1,t-1} - x_{2,t-1} + 5 \end{aligned}$$

- First we find the eigenvalues and eigenvectos of the matrix A. From Example 1. we know that there are two eigenvectors $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ (corresponding to $\lambda_1 = 3$) and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ (corresponding to $\lambda_2 = -2$). The decomposition of A is then $A = C\Lambda C^{-1}$ where
$$\Lambda = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix},$$
$$C = \begin{bmatrix} 4 & 1 \\ 1 & -1 \end{bmatrix},$$
and
$$C^{-1} = \begin{bmatrix} 0.2 & 0.2 \\ 0.2 & -0.8 \end{bmatrix}$$

- Defining $\bar{x}_t = C^{-1}x_t$ gives us the following system

$$\bar{x}_t = \Lambda \bar{x}_{t-1} + C^{-1}b$$

where

$$C^{-1}b = \begin{bmatrix} 0.2 & 0.2 \\ 0.2 & -0.8 \end{bmatrix} \begin{bmatrix} 10 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

or

$$\begin{aligned} \bar{x}_{1,t} &= 3\bar{x}_{1,t-1} + 3 \\ \bar{x}_{2,t} &= -2\bar{x}_{2,t-1} - 2 \end{aligned}$$

- Using the method developed earlier we can solve those equations. First note that in the steady state

$$\begin{aligned} \bar{x}_1 &= 3\bar{x}_1 + 3 \\ \bar{x}_2 &= -2\bar{x}_2 - 2 \end{aligned}$$

$$\begin{aligned} \bar{x}_1 &= -\frac{3}{2} \\ \bar{x}_2 &= -\frac{2}{3} \end{aligned}$$

Defining $\tilde{\bar{x}}_t = \bar{x}_t - \bar{x}$ we get

$$\begin{aligned} (\tilde{\bar{x}}_{1,t} + \bar{x}_1) &= 3(\tilde{\bar{x}}_{1,t-1} + \bar{x}_1) + 3 \\ (\tilde{\bar{x}}_{2,t} + \bar{x}_2) &= -2(\tilde{\bar{x}}_{2,t-1} + \bar{x}_2) - 2 \end{aligned}$$

Canceling out the steady state

$$\begin{aligned} \tilde{\bar{x}}_{1,t} &= 3\tilde{\bar{x}}_{1,t-1} \\ \tilde{\bar{x}}_{2,t} &= -2\tilde{\bar{x}}_{2,t-1} \end{aligned}$$

we get the following solution

$$\begin{aligned} \tilde{\bar{x}}_{1,t} &= A_1 3^t \\ \tilde{\bar{x}}_{2,t} &= A_2(-2)^t \end{aligned}$$

Therefore

$$\begin{aligned} \bar{x}_{1,t} &= -\frac{3}{2} + A_1 3^t \\ \bar{x}_{2,t} &= -\frac{2}{3} + A_2(-2)^t \end{aligned}$$

And since $x_0 = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$ we have $\bar{x}_0 = C^{-1}x_0 = \begin{bmatrix} 0.2 & 0.2 \\ 0.2 & -0.8 \end{bmatrix} \begin{bmatrix} 5 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$

$$\begin{aligned} 2 &= -\frac{3}{2} + A_1 3^0 \\ -3 &= -\frac{2}{3} + A_2(-2)^0 \end{aligned}$$

$$\begin{aligned} A_1 &= 2 + \frac{3}{2} = \frac{7}{2} \\ A_2 &= -3 + \frac{2}{3} = -\frac{7}{3} \end{aligned}$$

Substituting we get

$$\begin{aligned} \bar{x}_{1,t} &= -\frac{3}{2} + \frac{7}{2}3^t \\ \bar{x}_{2,t} &= -\frac{2}{3} - \frac{7}{3}(-2)^t \end{aligned}$$

- Finally, to get the solution of the initial problem we use the fact that since $\bar{x}_t = C^{-1}x_t$ we have $x_t = C\bar{x}_t$.

$$
\begin{aligned}
x_t &= C\bar{x}_t = \begin{bmatrix} 4 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -\frac{3}{2} + \frac{7}{2}3^t \\ -\frac{2}{3} - \frac{7}{3}(-2)^t \end{bmatrix} \\
&= \begin{bmatrix} 4(-\frac{3}{2} + \frac{7}{2}3^t) - \frac{2}{3} - \frac{7}{3}(-2)^t \\ -\frac{3}{2} + \frac{7}{2}3^t - (-\frac{2}{3} - \frac{7}{3}(-2)^t) \end{bmatrix} \\
&= \begin{bmatrix} -6 + 14 \cdot 3^t - \frac{2}{3} - \frac{7}{3}(-2)^t \\ -\frac{3}{2} + \frac{7}{2}3^t + \frac{2}{3} + \frac{7}{3}(-2)^t \end{bmatrix}
\end{aligned}
$$

which gives

$$
\begin{aligned}
x_{1,t} &= -\frac{20}{3} + 14 \cdot 3^t - \frac{7}{3}(-2)^t \\
x_{2,t} &= -\frac{5}{6} + \frac{7}{2}3^t + \frac{7}{3}(-2)^t
\end{aligned}
$$

Difference equation with no initial conditions

- Consider the following equation, with no initial condition

$$x_{t+1} = \rho x_t$$

assume the model cannot explode.

- Then we have:
  - $\rho > 1$, unique solution with $x_t = 0$ for all t.
  - $\rho = 1$, many solutions.
  - $\rho < 1$, many solutions.

BK method

Blanchard-Kahn method.

- Consider a system of equations

$$A \begin{bmatrix} x_{t+1} \\ E_t y_{t+1} \end{bmatrix} = B \begin{bmatrix} x_t \\ E_t y_t \end{bmatrix} + C\varepsilon_t$$

where x_t is a vector of n state variables (in case of RBC model (k, z)) and y_t is a vector of m control variables (in case of RBC model (c, l, w, r, y, x)) and ε_t is a vector of shocks.

- Note, we do have initial conditions for the state variables, but we do not have for the control variables.

- Assume A is non-singular (there are also methods for the case of singular A) and multiply both sides by A^{-1} to get

$$\begin{bmatrix} x_{t+1} \\ E_t y_{t+1} \end{bmatrix} = F \begin{bmatrix} x_t \\ E_t y_t \end{bmatrix} + G\varepsilon_t \tag{1}$$

where $F = A^{-1}B$ and $G = A^{-1}C$.

- Now use Jordan decomposition $F = H\Lambda H^{-1}$, where

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_{n+m} \end{bmatrix}$$

- Note that the eigenvalues and eigenvectors can be arranged in whatever order (as long as the kth column of H corresponds with the kth eigenvalue which occupies the (k,k) position of Λ). It is helpful to order the eigenvalues from smallest to largest in absolute value (note that if there are complex parts of the eigenvalues, order them by modulus, where the modulus the square root of the sum the squared non-complex and complex components; e.g. if $y = x + zi$, then the modulus is $\sqrt{x^2+z^2}$. If $z = 0$, the modulus is just the absolute value). Therefore,

$$|\lambda_1| < |\lambda_2| < ... < |\lambda_{n+m}|$$

- The model has unique solution if the number of unstable eigenvectors (greater than 1 in absolute value) of the system is exactly equal to the number of forward-looking (control) variables. In this case there is one solution, the equilibrium path is unique and the system exhibits saddle-path stability.

- Note, if there are too many stable roots then we have multiple equilibria. Or, if there are too many unstable roots, then we have no solution (paths are explosive and transversality condition is violated).

- If Blanchard-Kahn condition is satisfied we take

$$\begin{bmatrix} x_{t+1} \\ E_t y_{t+1} \end{bmatrix} = H\Lambda H^{-1} \begin{bmatrix} x_t \\ y_t \end{bmatrix} + G\varepsilon_t$$

and we multiply it by H^{-1} to get

$$H^{-1}\begin{bmatrix} x_{t+1} \\ E_t y_{t+1} \end{bmatrix} = \Lambda H^{-1} \begin{bmatrix} x_t \\ y_t \end{bmatrix} + H^{-1}G\varepsilon_t$$

or

$$\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}\begin{bmatrix} x_{t+1} \\ E_t y_{t+1} \end{bmatrix} = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix}\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}\begin{bmatrix} x_t \\ y_t \end{bmatrix} + \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} G\varepsilon_t \quad (2)$$

where Λ_1 consists of stable eigenvalues and Λ_2 consists of unstable eigenvalues and H^{-1} is respectively partitioned, define $H^{-1} = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}$.

- In order to simplify notation use the following notation

$$\begin{bmatrix} \tilde{x}_t \\ \tilde{y}_t \end{bmatrix} = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}\begin{bmatrix} x_t \\ y_t \end{bmatrix} \quad (3)$$

$$\begin{bmatrix} \tilde{G}_1 \\ \tilde{G}_2 \end{bmatrix} = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}\begin{bmatrix} G_1 \\ G_2 \end{bmatrix}$$

which gives us

$$\begin{bmatrix} \tilde{x}_{t+1} \\ E_t \tilde{y}_{t+1} \end{bmatrix} = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix}\begin{bmatrix} \tilde{x}_t \\ \tilde{y}_t \end{bmatrix} + \begin{bmatrix} \tilde{G}_1 \\ \tilde{G}_2 \end{bmatrix}\varepsilon_t \quad (4)$$

- First, we look at unstable part of the system

$$E_t\tilde{y}_{t+1} = \Lambda_2\tilde{y}_t + \tilde{G}_2\varepsilon_t$$

Solving for $\tilde{y}_t$

$$\tilde{y}_t = \Lambda_2^{-1}E_t\tilde{y}_{t+1} - \Lambda_2^{-1}\tilde{G}_2\varepsilon_t$$

Forwarding by one period

$$\tilde{y}_{t+1} = \Lambda_2^{-1}E_{t+1}\tilde{y}_{t+2} - \Lambda_2^{-1}\tilde{G}_2\varepsilon_{t+1}$$

Substituting back (note we use the law of iterative expectations $E_tE_{t+1}x_{t+2} = E_tx_{t+2}$)

$$\tilde{y}_t = \Lambda_2^{-1}E_t(\Lambda_2^{-1}E_{t+1}\tilde{y}_{t+2} - \Lambda_2^{-1}\tilde{G}_2\varepsilon_{t+1}) - \Lambda_2^{-1}\tilde{G}_2\varepsilon_t$$

$$\tilde{y}_t = \Lambda_2^{-2} E_t \tilde{y}_{t+2} - \Lambda_2^{-2} E_t(\tilde{G}_2 \varepsilon_{t+1}) - \Lambda_2^{-1} \tilde{G}_2 \varepsilon_t$$

Iterating it at infinity

$$\tilde{y}_t = -\Lambda_2^{-1} \tilde{G}_2 \varepsilon_t - \Lambda_2^{-2} E_t(\tilde{G}_2 \varepsilon_{t+1}) - \Lambda_2^{-3} E_t(\tilde{G}_2 \varepsilon_{t+2}) - ...$$

using the fact that $E_t \varepsilon_{t+1} = E_t \varepsilon_{t+2} = ... = 0$

$$\tilde{y}_t = -\Lambda_2^{-1} \tilde{G}_2 \varepsilon_t \tag{5}$$

which after plugging into (2) gives us y_t

$$\tilde{y}_t = H_{21} x_t + H_{22} y_t$$

$$H_{22} y_t = \tilde{y}_t - H_{21} x_t$$

$$y_t = H_{22}^{-1} \tilde{y}_t - H_{22}^{-1} H_{21} x_t$$

substituting for $\tilde{y}_t$ from (5) we get

$$y_t = -H_{22}^{-1} \Lambda_2^{-1} \tilde{G}_2 \varepsilon_t - H_{22}^{-1} H_{21} x_t \tag{6}$$

which gives us the values of control variables for given state variables and shocks.

- Next, we can come back to the stable part of (1)

$$x_{t+1} = F_{11} x_t + F_{22} y_t + G_1 \varepsilon_t$$

using (3)

$$x_{t+1} = F_{11} x_t + F_{22}(-H_{22}^{-1} \Lambda_2^{-1} \tilde{G}_2 \varepsilon_t - H_{22}^{-1} H_{21} x_t) + G_1 \varepsilon_t$$

which gives us the values of next period state variables given values of (current period) state variables and shocks.

Summary

- A little bit about solving difference equations.
- Blanchard-Kahn method.